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NEW ASPECTS OF FRACTAL ON HIGHER DIMENSIONS

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List of publications

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1. Md. N. Akhtar and **A. Hossain**, Stereographic metric and dimensions of fractals on the sphere, Results in Mathematics **77**(6), 1–31 (2022). https://doi.org/10.1007/s00025-022-01745-x

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Abstract

The primary focus of this current research is centered around the formulation of an iterated function system (IFS) on the sphere (Euclidean space), as well as the development of an IFS on the projective plane (non-Euclidean space). To begin with, we established an IFS on the sphere and defined a metric (known as a stereographic metric) on it to demonstrate the contractive properties of this IFS. We described some of the topological and geometric properties of the attractor of such an IFS. We investigated relations between various dimensions of a non-empty compact subset of $\mathbb{R}^2$ as well as a non-empty compact subset of the sphere with respect to the various metrics. Also, we compared the dimensions of a non-empty compact subset on the sphere and its projection on the plane with respect to different metrics.
After that, we considered the projective plane, which avoids a hyperplane and provides a decomposition on it. We constructed an IFS on the projective plane whose attractor is the graph of an affine fractal function (which is known as real projective fractal interpolation function (RPFIF)), on the projective interval. To do this, we defined a new metric on it, which we used to show that the IFS is contractive. Also, we estimated the fractal dimension of the graph of a RPFIF. Also considering a suitable IFS on the real projective plane, we constructed a non-affine fractal function on it. Some fractal versions of the classical approximation results on the projective plane are proved. Furthermore, we established an IFS on the dual space of the projective plane and demonstrated that the attractor of such an IFS is also the graph of a non-affine fractal function.
We are currently working on the quantization dimensions of Cantor-like sets on the projective line.

Chapter 1

Introduction and Motivation

Fractals are geometric shapes that can exhibit rough or non-smooth characteristics, with each individual component sharing the same statistical properties as the entire structure. Mandelbrot (24), coined the term 'fractal,' derived from 'fractured' or 'broken,' to expand the concept of theoretical fractional dimensions to describe natural geometric patterns. Mandelbrot set, snowflakes, Sierpiński triangle, Cantor set, Julia set and Barnsley fern are a few examples of fractal sets. In the field of mathematics, an iterated function system (IFS) is employed as a method for generating fractal structures. Hutchinson (23), Mandelbrot (24), applied this method to create self-similar fractal patterns. Barnsley's work helped to make this formalism widely popular (6; 8). Numerous studies have been conducted in the literature on fractal construction (2; 4; 7; 9; 17; 25; 27). The essence of fractal geometry lies at the core of dimension theory. The fractal dimension serves as a quantifiable index (which may not necessarily be a whole number) used to characterize the complexity of fractal patterns or sets (1; 3; 10; 24; 20).

Definition 1.0.1 (Hausdörff metric) *Let (X, d) be a metric space, and $\mathscr{H}(X)$ denotes the space of all non-empty compact subsets of X. Then the Hausdörff metric H_d in $\mathscr{H}(X)$ is defined by*

$$H_d(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \quad \sup_{y\in B}\inf_{x\in A} d(x, y)\right\} \quad \textit{for all } A, B \in \mathscr{H}(X).$$

Definition 1.0.2 (Iterated function system (IFS)) *Let (X, d) be a complete metric space and $N_m = \{1, 2, \ldots, m\}$. If $W_i : X \to X$, $i \in N_m$ are the continuous functions, then $\mathscr{W} = \{X; W_i : i \in N_m\}$ is called an **iterated function system** (see (8)). The system is called hyperbolic if each function $W_i : X \to X$ is contractive with contraction factor $0 \leq c_i < 1$, $i \in N_m$.*

The **Hutchinson operator** $W : \mathscr{H}(X) \to \mathscr{H}(X)$, is defined by

$$W(B) = \bigcup_{i=1}^{m} W_i(B) \quad \text{for all } B \in \mathscr{H}(X).$$

Definition 1.0.3 *(see (6)) A compact subset F of (X, d) is called an **attractor** of the IFS $\mathscr{W} = \{X; W_i : i \in N_m\}$ if*

1. $W(F) = F$ *and*

2. *there exists an open subset* U *of* X *such that* $F \subset U$ *and*

$$\lim_{k\to\infty} W^k(B) = F \quad \text{for all } B \in \mathscr{H}(U),$$

where the limit is with respect to the Hausdörff metric H_d *on* $\mathscr{H}(X)$.

Note 1.0.1 *The largest open set* U *in Definition 1.0.3 is known as the* ***basin*** *of attraction for the attractor* F *of the IFS* $\mathscr{W}$ *and is denoted by* $B(F)$.

Definition 1.0.4 *A fast basin of the IFS* $\mathscr{W}$ *with an attractor* F *is the following set*

$$\hat{B}(F) = \{x \in X : \exists \quad k \geq 0 \text{ such that } W^k(x) \cap F \neq \emptyset\}.$$

Definition 1.0.5 *An IFS* $\mathscr{W}$ *is said to be invertible if for all* $W \in \mathscr{W}$, $W : X \to X$ *is a homeomorphism.*

For a finite word $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \{1, 2, \ldots, N\}^k$, define

$$W_{\sigma_1\sigma_2\cdots\sigma_k} = W_{\sigma_1} \circ W_{\sigma_2} \circ \cdots \circ W_{\sigma_k} \quad \text{and} \quad W_\emptyset = Id_X,$$

where $\emptyset$ is the empty (zero-length) word and Id_X is the identity map on X. Also, given an infinite word $\sigma = (\sigma_1, \sigma_2, \ldots) \in \{1, 2, \ldots, N\}^\infty$, we write $\sigma|k = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \{1, 2, \ldots, N\}^k$ and define $\sigma|0 = \emptyset$. Also, if the IFS $\mathscr{W}$ is invertible, define

$$W^{-1}_{\sigma|k} = W^{-1}_{\sigma_1} \circ W^{-1}_{\sigma_2} \circ \cdots \circ W^{-1}_{\sigma_k}$$

and

$$W^{-1}_{\sigma|k}(C) = \left\{x \in X : W_{\sigma_k} \circ W_{\sigma_{k-1}} \circ \cdots \circ W_{\sigma_1}(x) \in C\right\}$$

for all $C \subset X$.

Definition 1.0.6 *A fractal continuation of an attractor* F *of an invertible IFS* $\mathscr{W}$ *along the infinite word* $\sigma = (\sigma_1, \sigma_2, \ldots)$ *is the ascending union*

$$\hat{B}_F(\sigma) = \bigcup_{k\geq 0} W^{-1}_{\sigma|k}(F). \tag{1.1}$$

Definition 1.0.7 *An attractor* F *of the IFS* $\mathscr{W}$ *is said to be point-fibred (w.r.t.* $\mathscr{W}$*) if* $\lim_{k\to\infty} W_{\sigma|k}(C)$ *exists and is a singleton subset of* F*, where* C *is a non-empty compact subset of* $B(F)$.

A fractal interpolation function (FIF) can be characterized as a continuous function that interpolates particular data points and possesses a graph that serves as the attractor, forming a fractal set via an IFS. Barnsley initially introduced fractal functions as continuous functions that interpolate a provided set of data points (7). After that, Navascués (25; 26; 27), introduced a class of fractal functions known as α-fractal functions as a perturbation of any continuous functions. Most of the authors studied fractal interpolation on the Euclidean spaces (8; 9; 12; 15; 26). Recently, Barnsley et al. (11), studied IFSs on the real projective plane.

Definition 1.0.8 (Real projective space) *Given an Euclidean space $\mathbb{R}^{n+1}$, the real projective space associated with $\mathbb{R}^{n+1}$ is the set $\mathbb{RP}^n$ of one dimensional subspaces or (vector) lines in $\mathbb{R}^{n+1}$.*

Definition 1.0.9 (Hyperplane) *If $p, q \in \mathbb{RP}^n$ have the homogeneous coordinates $(p_1, p_2, \ldots, p_{n+1})$ and $(q_1, q_2, \ldots, q_{n+1})$ respectively, and $\sum_{k=1}^{n+1} p_k q_k = 0$, then we say that p is orthogonal to q, and write $p \perp q$. A* **hyperplane** *in $\mathbb{RP}^n$ is a set of the form*

$$\mathbb{H}_p = \left\{ q \in \mathbb{RP}^n : p \perp q \right\} \subseteq \mathbb{RP}^n$$

for some $p \in \mathbb{RP}^n$.

Definition 1.0.10 (see (11)) *A set $\mathbb{K} \subseteq \mathbb{RP}^n$ is said to* **avoid a hyperplane** *if there exists a hyperplane $\mathbb{H}_p \subseteq \mathbb{RP}^n$ such that $\mathbb{H}_p \cap \mathbb{K} = \emptyset$.*

Definition 1.0.11 (Projective IFS) *An IFS $\mathscr{W} = \{\mathbb{RP}^n; W_n : n = 1, 2, \ldots, N\}$ is said to be* ***real projective iterated function system*** *or RPIFS if each W_n is a projective transformations on $\mathbb{RP}^n$ (11).*

Fractal function:

Let $I = [a, b]$ and $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be a partition of I. Let $\{(x_n, y_n) \in \mathbb{R}^2 : n = 0, 1, 2, \ldots, N\}$ be the given interpolation points in $\mathbb{R}^2$. Set $I_n = [x_{n-1}, x_n]$ for $n = 1, 2, \ldots, N$. Suppose $L_n : I \to I_n$ are contraction homeomorphisms such that

$$L_n(x_0) = x_{n-1},\ L_n(x_N) = x_n, \tag{1.2}$$

$$|L_n(x) - L_n(x')| \leq c_n |x - x'| \quad \text{for all } x, x' \in I, \text{ for some } 0 \leq c_n < 1. \tag{1.3}$$

Further, assume that $F_n : I \times \mathbb{R} \to \mathbb{R}$ are continuous maps satisfying

$$F_n(x_0, y_0) = y_{n-1},\ F_n(x_N, y_N) = y_n, \tag{1.4}$$

$$\begin{aligned} |F_n(x, y) - F_n(x', y)| &\leq a_n |x - x'| \quad \text{for all } x, x' \in I, \\ |F_n(x, y) - F_n(x, y')| &\leq b_n |y - y'| \quad \text{for all } y, y' \in \mathbb{R}, \end{aligned}$$

for some $a_n, b_n \in (-1, 1)$. Define the functions $W_n : I \times \mathbb{R} \to I_n \times \mathbb{R}$ by

$$W_n(x, y) = (L_n(x), F_n(x, y)). \tag{1.5}$$

The maps W_n satisfy the join up condition

$$W_n(x_0, y_0) = (x_{n-1}, y_{n-1}), \quad W_n(x_N, y_N) = (x_n, y_n) \quad \text{for all } n = 1, 2, \ldots, N. \tag{1.6}$$

The following is a fundamental result in the theory of fractal interpolation functions.

Theorem 1.0.1 (Barnsley (7)) *The IFS $\{(I \times \mathbb{R}; W_n) : n = 1, 2, \ldots, N\}$ has unique attractor $G(g)$ which is the graph of a continuous function $g : I \to \mathbb{R}$ satisfying $g(x_n) = y_n$ for all $n = 0, 1, \ldots, N$.*

The function g is called the **fractal interpolation function** (FIF) corresponding to the data set $\{(x_n, y_n) \in \mathbb{R}^2 : n = 0, 1, 2, \ldots, N\}$.
In mathematics, quantization is the process of mapping input values from a large set to output values in a smaller set. The aim of quantization for probability measures is to approximate a given probability measure with a discrete probability measure having finite support. In modern science, quantization is very useful for signal processing, image processing, etc. (14; 33; 35). Recently, Roychowdhury determined the optimal sets of n-means and the n-th quantization errors for the self-similar measure P generated by an IFS in his articles (31; 32).

In this thesis, primarily our aim is to elucidate our discoveries concerning certain characteristics of IFSs and their attractors on the sphere through the utilization of stereographic maps. Secondly, we constructed the RPFIF on the projective plane and estimated the fractal dimension of the graph associated with certain RPFIF. Also, we constructed a non-affine fractal function on the real projective plane by considering a suitable IFS on it. After that, we study the approximation properties of the constructed non-affine RPFF on the projective plane. We considered an IFS on the projective line whose attractor is a Cantor-like set on it. We defined an invariant probability measure P generated by this IFS. Finally, we provided an upper bound for the n-th quantization error of order 2 for the probability measure P. The contents of the proposed thesis are divided into six chapters as follows:

Chapter-1: Introduction and Motivation

Chapter-2: Iterated Function Systems and Dimensions of Fractals on the Sphere

Chapter-3: Fractal Interpolation Function on the Real Projective Plane

Chapter-4: Fractal Dimensions of the Real Projective Fractal Interpolation Functions

Chapter-5: Non-Affine Fractal Function on the Real Projective Plane

Chapter-6: Quantization of Cantor-like set on the Projective Line

Chapter 2

Iterated Function Systems and Dimensions of Fractals on the Sphere

In this chapter, first, we consider an IFS on the sphere $\mathbf{S} \setminus N$ and see the existence of its attractor. Also, we characterize some topological and geometrical properties of the attractor.

2.1 Iterated function system on the sphere

Let $\mathbb{R}^2_z = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, $\mathbf{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - p)^2 = 1\}$, $p \geq 1$ be an sphere and N be the north pole $(0, 0, p + 1)$ of it. The IFS on the sphere consists of the transformations that are constructed by using the stereographic projection map P and its inverse map P^{-1}. To show the contractivity of this IFS, we defined a new metric on the sphere which we are calling as **stereographic metric** d_S. Also, we consider the **chordal metric** d_C on the plane (34).
In the sequel, we will use the same notation P^{-1} and P for the restrictions on the compact subsets $X \subset \mathbb{R}^2_z$ and $Y \subset \mathbf{S} \setminus N$, respectively.
The following theorem proves the existence of the attractor of an IFS on the sphere.

Theorem 2.1.1 *Let $X \subset \mathbb{R}^2_z$ be compact and $\mathscr{W} = \{X; W_i : i \in N_m\}$ be an IFS with attractor F' with respect to the Hausdörff metric H_{d_2}. Then $\mathscr{T} = \{P^{-1}(X); T_i := P^{-1}W_iP : i \in N_m\}$ is an IFS with attractor $P^{-1}(F')$ with respect to the Hausdörff metric H_{d_S}. Conversely if $Y \subset \boldsymbol{S} \setminus N$, is compact and $\mathscr{T} = \{Y; T_i : i \in N_m\}$ is an IFS with attractor F with respect to the Hausdörff metric H_{d_3}, then $\mathscr{W} = \{P(Y); W_i := PT_iP^{-1} : i \in N_m\}$ is an IFS with attractor $P(F)$ with respect to the Hausdörff metric H_{d_C}.*

Theorem 2.1.2 *If F is the attractor of the IFS $\mathscr{W}$ and $P^{-1}(F)$ is the attractor of the IFS $\mathscr{T}$, then the following hold.*

1. *If $B(F)$ is the basin of attraction of the attractor F, then $P^{-1}(B(F))$ is the basin of attraction of the attractor $P^{-1}(F)$. That is*
   $B(P^{-1}(F)) = P^{-1}(B(F))$.
2. *If $\hat{B}(F)$ is the fast basin of F, then $P^{-1}(\hat{B}(F))$ is the fast basin of $P^{-1}(F)$. That is $\hat{B}(P^{-1}(F)) = P^{-1}(\hat{B}(F))$.*

3. *If F is point-fiberd attractor (w.r.t. $\mathscr{W}$), then $P^{-1}(F)$ is also point-fibred attractor (w.r.t. $\mathscr{T}$).*

4. *If $\hat{B}_F(\sigma)$ is a fractal continuation of the attractor F of an invertible IFS $\mathscr{W}$ along the infinite word $\sigma = (\sigma_1, \sigma_2, \ldots)$, then $P^{-1}(\hat{B}_F(\sigma))$ is also a fractal continuation of the attractor $P^{-1}(F)$ along the infinite word $\sigma = (\sigma_1, \sigma_2, \ldots)$. That is $\hat{B}_{P^{-1}(F)}(\sigma) = P^{-1}(\hat{B}_F(\sigma))$.*

2.2 Comparative results on dimensions

In this section, we investigate the comparison of dimensions (with respect to different metrics) of a nonempty compact subset F of $\mathbf{S} \setminus N$. A similar investigation has been made for a nonempty compact subset K of $\mathbb{R}^2_z$. Also, we investigate the comparison in between the dimensions (with respect to different metrics) of a nonempty compact subset F of $\mathbf{S} \setminus N$ and its projection $F' = P(F)$ on $\mathbb{R}^2_z$.

2.2.1 Hausdörff dimension

Theorem 2.2.1 *Let F and K be the compact subsets of $\boldsymbol{S} \setminus N$ and $\mathbb{R}^2_z$ respectively. Then*

1. $\dim_H(F, d_3) \leq \dim_H(F, d_S)$.
2. $\dim_H(K, d_C) \leq \dim_H(K, d_2)$.

Theorem 2.2.2 *Let F be a compact subset of $\boldsymbol{S} \setminus N$ and $F' = P(F)$. Then*

1. $\dim_H(F, d_3) = \dim_H(F', d_C)$.
2. $\dim_H(F, d_S) = \dim_H(F', d_2)$.
3. $\dim_H(F, d_3) \leq \dim_H(F', d_2)$.

2.2.2 Box dimension

Theorem 2.2.3 *Let F and K be the compact subsets of $\boldsymbol{S} \setminus N$ and $\mathbb{R}^2_z$ respectively. Then*

1. $\dim_B(F, d_3) \leq \dim_B(F, d_S)$.
2. $\dim_B(K, d_C) \leq \dim_B(K, d_2)$.

Theorem 2.2.4 *Let F be a compact subset of $\boldsymbol{S} \setminus N$ and $F' = P(F)$. Then*

1. $\dim_B(F, d_3) = \dim_B(F', d_C)$.
2. $\dim_B(F, d_S) = \dim_B(F', d_2)$.
3. $\dim_B(F, d_3) \leq \dim_B(F', d_2)$.

2.2.3 Assouad dimension

Theorem 2.2.5 *Let F and K be the compact subsets of $\boldsymbol{S} \setminus N$ and $\mathbb{R}^2_z$ respectively. Then*

1. $\dim_A(F, d_3) \leq \dim_A(F, d_S)$.
2. $\dim_A(K, d_C) \leq \dim_A(K, d_2)$.

Theorem 2.2.6 *Let F be a subset of $\boldsymbol{S} \setminus N$ and $F' = P(F)$. Then*

1. $\dim_A(F, d_3) = \dim_A(F', d_C)$.
2. $\dim_A(F, d_S) = \dim_A(F', d_2)$.

The following table summarises the possible relationships between the Hausdörff (or Box) dimension of the set $F \subset \mathrm{S} \setminus N$ and its projection $F' = P(F) \subset \mathbb{R}^2_z$ with respect to the Euclidean, stereographic and chordal metrics. Here "$\dim(F, d)$" denotes the Hausdörff (or Box) dimension with respect to the metric d.

Table 2.1:

$\dim(F, d_3)$	$=$	$\leq$	$\leq$	$=$
$\dim(F, d_S)$	$\geq$	$=$	$=$	$\geq$
$\dim(F', d_2)$	$\geq$	$=$	$=$	$\geq$
$\dim(F', d_C)$	$=$	$\leq$	$\leq$	$=$
	$\dim(F, d_3)$	$\dim(F, d_S)$	$\dim(F', d_2)$	$\dim(F', d_C)$

2.3 Conclusion and Open Issues

In this chapter, we considered an IFS on the plane and equivalently on the sphere via the stereographic projection map and its inverse map. The construction of a contractive IFS on the sphere and the study of the topological and geometric properties of this IFS and its attractor are not that much explored. For the reasons of convenience to study an IFS on the sphere, we introduced a new metric (stereographic metric) which is equivalent to the Euclidean metric on the sphere. We proved that the contractivity of an IFS on the plane with respect to the usual metric is equivalent to the contractivity of the corresponding IFS on the sphere with respect to the stereographic metric. Here, we investigated various dimensions of a non-empty compact subset on the sphere and its projection on the plane with respect to different metrics.

In the definition of the sphere, we begin with setting $p \geq 1$, and subsequently explore the comparison of different dimensions within a non-empty compact subset on the sphere and its projection onto the plane, examining their behavior under various metrics.
The first question is.

Question 1: If $p = 0$, that is if the center of the sphere is the origin, then what are the relations between the different dimensions within a non-empty compact subset on the sphere and its projection onto the plane with respect to the various metrics?
The next question is.

Question 2: Is it possible to define an Iterated Function System (IFS) on the sphere in such a way that its attractor is the graph of a continuous (fractal) function?

Chapter 3

Fractal Interpolation Function on the Real Projective Plane

In this chapter, we introduce a linear structure and a metric on projective space. We consider an iterated function system that gives rise to a fractal interpolation function of a set of data points.

3.1 Decomposition of the projective plane which avoids a hyperplane

Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical basis of $\mathbb{R}^3$. Then $\mathbb{H}_{e_i}$ is the hyperplane perpendicular to e_i for $i = 1,2,3$. In the sequel, we consider the space $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ in particular and define two operations $\oplus$ and $\odot$ as follows. For all $(x : y : z), (x' : y' : z') \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ and for all $a \in \mathbb{R}$,

$$\begin{aligned}(x : y : z) \oplus (x' : y' : z') &:= (xz' + x'z : yz' + y'z : zz'),\\ a \odot (x : y : z) &:= (ax : ay : z).\end{aligned}$$

The space $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ forms a vector space over $\mathbb{R}$ with respect to the above defined operations $\oplus$ and $\odot$. If $(x_1 : y_1 : z_1), (x_2 : y_2 : z_2) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, define $(x_1 : y_1 : z_1) \ominus (x_2 : y_2 : z_2) = (x_1z_2 - x_2z_1 : y_1z_2 - y_2z_1 : z_1z_2)$. So, each element $(x : y : z)$ in $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ can be expressed as a sum of two of its elements namely, $(x : 0 : z)$ and $(0 : y : z)$. That is $(x : y : z) = (x : 0 : z) \oplus (0 : y : z)$. Let $\mathbb{H}_{10} := \big\{(x : 0 : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\big\}$ and $\mathbb{H}_{01} := \big\{(0 : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\big\}$. Then $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ can be expressed as

$$\mathbb{RP}^2 \setminus \mathbb{H}_{e_3} = \mathbb{H}_{10} \oplus \mathbb{H}_{01}. \tag{3.1}$$

We define the real projective norm on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ as follows:

$$\|(x : y : z)\|_{\mathbb{P}} := \frac{\sqrt{x^2 + y^2}}{|z|} \tag{3.2}$$

for all $(x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. The **projective metric**, is denoted by $d_{\mathbb{P}}$ and given by

$$d_{\mathbb{P}}\big((x : y : z), (x' : y' : z')\big) := \|(x : y : z) \ominus (x' : y' : z')\|_{\mathbb{P}}. \tag{3.3}$$

For $(x_1 : 0 : z_1), (x_2 : 0 : z_2) \in \mathbb{H}_{10}$, we say that $(x_1 : 0 : z_1) \preceq (x_2 : 0 : z_2)$, if and only if $x_1z_2 \leq x_2z_1$.

Definition 3.1.1 (Projective interval on $\mathbb{H}_{10}$) *Let $(a_1 : 0 : c_1), (a_2 : 0 : c_2) \in \mathbb{H}_{10}$ be such that $(a_1 : 0 : c_1) \prec (a_2 : 0 : c_2)$. Then the projective interval on $\mathbb{H}_{10}$, is denoted by $\mathbb{P}_{\mathbb{I}\times\{0\}}$, and is defined by*

$$\mathbb{P}_{\mathbb{I}\times\{0\}} := \Big\{(x : 0 : z) \in \mathbb{H}_{10} : \ (a_1 : 0 : c_1) \preceq (x : 0 : z) \preceq (a_2 : 0 : c_2)\Big\}.$$

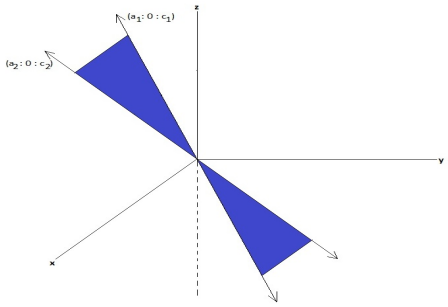

Figure 3.1: Projective interval.

Let

$$\mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}] = \Big\{f : f : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{H}_{01} \text{ is continuous}\Big\}. \tag{3.4}$$

If $f \in \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$, define $\|f\|_{\mathbb{P}\infty} := \sup\{\|f(x : 0 : z)\|_{\mathbb{P}} : \ (x : 0 : z) \in \mathbb{P}_{\mathbb{I}\times\{0\}}\}$. The space $\big(\mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}], \|\cdot\|_{\mathbb{P}\infty}\big)$ is a complete normed linear space.

3.2 Real projective fractal interpolation function

Following is the setup for the construction of a real projective fractal interpolation function. Let $N \geq 2$ and $\big\{(x_n : y_n : z_n) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : \ n = 0, 1, \ldots, N\big\}$ be a data set in $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ such that $x_n z_{n+1} < x_{n+1} z_n$ for $n = 0, 1, \ldots, N-1$. Let $\mathbb{P}_{\mathbb{I}\times\{0\}} := \Big\{(x : 0 : z) \in \mathbb{H}_{10} : \ (x_0 : 0 : z_0) \preceq (x : 0 : z) \preceq (x_N : 0 : z_N)\Big\}$ and $\mathbb{P}_{\mathbb{I}_n\times\{0\}} := \Big\{(x : 0 : z) \in \mathbb{H}_{10} : \ (x_{n-1} : 0 : z_{n-1}) \preceq (x : 0 : z) \preceq (x_n : 0 : z_n)\Big\}$. For $n = 1, 2, \ldots, N$, consider the transformations $L_n : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{P}_{\mathbb{I}_n\times\{0\}}$ given by $L_n(x : 0 : z) = (a_n x + b_n z : 0 : z)$ such that

$$L_n(x_0 : 0 : z_0) = (x_{n-1} : 0 : z_{n-1}) \text{ and } L_n(x_N : 0 : z_N) = (x_n : 0 : z_n), \tag{3.5}$$

where $a_n, b_n \in \mathbb{R}$.
Also, for $n = 1, 2, \ldots, N$, consider the continuous maps $F_n : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \to \mathbb{H}_{01}$ given by

$$F_n(x : y : z) = \big(0 : c_n x + d_n y + f_n z : z\big) \tag{3.6}$$

such that

$$F_n(x_0 : y_0 : z_0) = (0 : y_{n-1} : z_{n-1}) \quad \text{and} \quad F_n(x_N : y_N : z_N) = (0 : y_n : z_n), \tag{3.7}$$

where $c_n, d_n, f_n \in \mathbb{R}$. Define the functions $W_n : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \to \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ by

$$W_n(x : y : z) = L_n(x : 0 : z) \oplus F_n(x : y : z). \tag{3.8}$$

Since the space $\big(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}, d_{\mathbb{P}}\big)$ is complete. So, the RPIFS $\big\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n : \ n = 1, 2, \ldots, N\big\}$ has an unique attractor, say G. Now, we show that G is the graph of a continuous function from $\mathbb{P}_{\mathbb{I}\times\{0\}}$ to $\mathbb{H}_{01}$. Let

$$\mathscr{F} := \big\{f \in \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}] : \ f(x_0 : 0 : z_0) = (0 : y_0 : z_0),\ f(x_N : 0 : z_N) = (0 : y_N : z_N)\big\}.$$

Then $\mathscr{F}$ is a closed subset of $\big(\mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}], \|\cdot\|_{\mathbb{P}\infty}\big)$. So, $\mathscr{F}$ is complete. Finally, we define a **real projective Read-Bajraktarevic**-operator (RPRB) $\mathcal{T} : \mathscr{F} \to \mathscr{F}$, as follows.

$$(\mathcal{T}f)(x : 0 : z) := F_n\big(L_n^{-1}(x : 0 : z) \oplus f \circ L_n^{-1}(x : 0 : z)\big). \tag{3.9}$$

Theorem 3.2.1 *The RPRB-operator $\mathcal{T}$ is well defined and contractive on $\big(\mathscr{F}, \|\cdot\|_{\mathbb{P}\infty}\big)$.*

Since $\big(\mathscr{F}, \|\cdot\|_{\mathbb{P}\infty}\big)$ is complete, by Banach fixed point theorem $\mathcal{T}$ has an unique fixed point $\mathbf{f}$ in $\mathscr{F}$. We call $\mathbf{f}$ as the **real projective fractal interpolation function** (RPFIF) corresponding to the RPIFS $\big\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n : \ n = 1, 2, \ldots, N\big\}$.

Theorem 3.2.2 *The graph of the function $\mathbf{f}$ is the attractor of the RPIFS $\big\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n : \ n = 1, 2, \ldots, N\big\}$. That is $G = graph(\mathbf{f})$.*

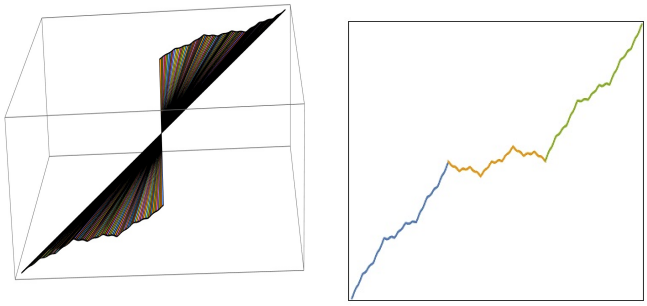

Figure 3.2: Graphs of the RPFIF and its corresponding self-affine FIF at level $z = z_0$ ($\neq 0$).

3.3 Conclusion and Open Issues

In this chapter, a method is developed to construct a RPFIF for a given data set on the real projective plane $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. The advantage of the construction of such a RPFIF is that it is an infinite fractal (in $\mathbb{R}^3$) consisting of self-affine fractal interpolation functions (which are similar to each other up to contraction) giving thereby a choice of large flexibility in approximating functions.

- Here, we considered the space $\mathbb{RP}^2$ for notational simplicity only. One may consider projective space $\mathbb{RP}^n$ with more dimensions and deal with the bivariate case.

- In Section 3.2, instead of beginning with scalar d_n, the idea can be extended to the model which considers d_n depending on a variable(s).
- Projective tiling has many real-world applications. So, one may study about the fractal projective tiling on a projective space.

Chapter 4

Fractal Dimensions of the Real Projective Fractal Interpolation Functions

In the literature, the concept of several dimensions of the fractal sets with respect to the Euclidean distance on the plane has been largely treated (see for instance (1; 5; 8; 9; 16; 17; 18; 20)). Hossain et al. (22) introduced the **real projective fractal interpolation function** (**RPFIF**) by considering a **real projective iterated function system** (**RPIFS**) on the projective plane.
In this chapter, on the basis of these concepts, we estimate the fractal dimension of the graph associated with a certain RPFIF.
One of the interesting features of projective geometry is that of duality. Dual space is the collection of all hyperplanes of a projective space. At the end, a construction for dual IFS is provided.

4.1 Fractal dimension of the graph of a RPFIF

In this section, we estimate the fractal dimension of the graph G of a RPFIF when the interpolation points are equispaced.

4.1.1 Fractal dimension of the graph of a RPFIF as a subset of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$

Let $\mathbb{P}_{I\times\{0\}} = \left\{(x:0:z) \in \mathbb{H}_{10} : \ (0:0:1) \preceq (x:0:z) \preceq (1:0:1)\right\}$ and $X = \mathbb{P}_{I\times\{0\}} \oplus \mathbb{H}_{01}$ and let $N \geq 2$, and $\{(x_i : y_i : z_i) : i = 0, 1, 2, \ldots, N\}$ be the $N+1$ interpolation points in X such that $(x_i : 0 : z_i) \ominus (x_{i-1} : 0 : z_{i-1}) = (1 : 0 : N)$. Define a hyperbolic RPIFS $\{(X; W_i) : i = 1, 2, \ldots, N\}$ such that

$$W_i \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & i-1 \\ Nc_i & Nd_i & Nk_i \\ 0 & 0 & N \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4.1}$$

If the parameters d_i are such that $|d_i| < 1$, then the maps W_i, $i \in \{1, 2, \ldots, N\}$ are contractive. Hence the above RPIFS possesses a unique attractor which is the graph of a

continuous function $\mathbf{f} : \mathbb{P}_{I\times\{0\}} \to \mathbb{H}_{01}$ passing through the data points $(x_i : 0 : z_i)$, $i \in \{0, 1, 2, \ldots, N\}$. Let G be the graph of $\mathbf{f}$. That is

$$G = \left\{(x : 0 : z) \oplus \mathbf{f}(x : 0 : z) : \ (x : 0 : z) \in \mathbb{P}_{I\times\{0\}}\right\}.$$

Theorem 4.1.1 *If $G = graph(\mathbf{f})$, $\mu = \sum_{i=1}^{N} |d_i| > 1$ and the interpolation points are not collinear, then $\dim_B(G) = 1 + \log_N \mu$; otherwise $\dim_B(G) = 1$.*

4.1.2 Fractal dimension of the graph of a RPFIF as a subset of $\mathbb{R}^3$

Theorem 4.1.2 *If $\mu > 1$ and the interpolation points are not co-planar, then $\dim_B(G) = 2 + \log_N \mu$; otherwise $\dim_B(G) = 2$.*

4.2 Dual RPIFS

Definition 4.2.1 *Let $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ denote the set of all hyperplanes of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, or equivalently $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} = \left\{p^\perp : p \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\right\}$. Then the space $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ is said to be the dual space of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. The addition on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ is induced from the addition on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. That is $p^\perp \oplus q^\perp = (p \oplus q)^\perp$. The dual space is endowed with a metric $\widehat{d}_{\mathbb{P}}$ defined by*

$$\widehat{d}_{\mathbb{P}}(p^\perp, q^\perp) := d_{\mathbb{P}}(p, q) \quad \text{for all } p^\perp, q^\perp \in \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}. \tag{4.2}$$

The map $Q : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \to \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ defined by $Q(p) = p^\perp$ is called the duality map.

Let $\widehat{\mathbb{H}}_{10} = Q(\mathbb{H}_{10})$ and $\widehat{\mathbb{H}}_{01} = Q(\mathbb{H}_{01})$. Now, for $x \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, x can be written as $x = x_1 \oplus x_2$, where $x_1 \in \mathbb{H}_{10}$ and $x_2 \in \mathbb{H}_{01}$. Then from the definition of the addition on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$, $x^\perp = (x_1 \oplus x_2)^\perp = x_1^\perp \oplus x_2^\perp$. So, $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ can be expressed as

$$\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} = \widehat{\mathbb{H}}_{10} \oplus \widehat{\mathbb{H}}_{01}. \tag{4.3}$$

Now, for a given data set $\{(x_n : y_n : z_n) : \ n = 0, 1, \ldots, N\}$ on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, we can extend L_n and F_n, which are defined in (3.5) and (3.6) respectively, as follows

$$\widehat{L}_n : \widehat{\mathbb{P}}_{I\times\{0\}} \to \widehat{\mathbb{P}}_{I_n\times\{0\}}, \quad \widehat{F}_n : \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} \to \widehat{\mathbb{H}}_{01} \tag{4.4}$$

such that $\widehat{L}_n\big((x : 0 : z)^\perp\big) = \big(L_n(x : 0 : z)\big)^\perp$ and $\widehat{F}_n\big((x : y : z)^\perp\big) = \big(F_n(x : y : z)\big)^\perp$, where $\widehat{\mathbb{P}}_{I\times\{0\}} = Q\big(\mathbb{P}_{I\times\{0\}}\big)$ and $\widehat{\mathbb{P}}_{I_n\times\{0\}} = Q\big(\mathbb{P}_{I_n\times\{0\}}\big)$. Define $\widehat{W}_n : \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} \to \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ such that

$$\widehat{W}_n\big((x : y : z)^\perp\big) = \widehat{L}_n\big((x : 0 : z)^\perp\big) \oplus \widehat{F}_n\big((x : y : z)^\perp\big). \tag{4.5}$$

Definition 4.2.2 *The collection $\left\{\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}; \widehat{W}_n : \ n = 1, 2, \ldots, N\right\}$ is said to be a dual RPIFS.*

Figure 4.1 represents the attractor of a dual RPIFS.

Conjecture 1 *If G is the graph of the RPFIF corresponding to the RPIFS $\mathscr{W} = \left\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n : \ n = 1, 2, \ldots, N\right\}$ given by (3.8), then there exists an attractor $\widehat{G}$ corresponding to the dual RPIFS $\widehat{\mathscr{W}} = \left\{\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}; \widehat{W}_n : \ n = 1, 2, \ldots, N\right\}$ such that $\widehat{G}$ is also the graph of a self-referential function. Moreover, $\widehat{G} = \left\{p^\perp : p \in G\right\}$.*

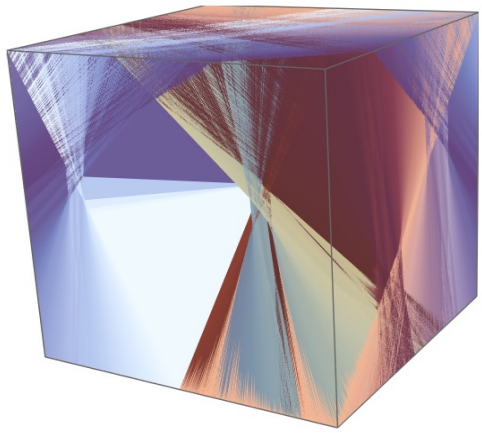

Figure 4.1: Attractor of a dual RPIFS.

4.3 Conclusions

In this article, we have estimated the fractal dimension of the graph of a RPFIF on the real projective plane which has a linear structure. Since the graph of the RPFIF can viewed as a subset of $\mathbb{RP}^2$ as well as a subset of $\mathbb{R}^3$, we have calculated the dimensions for both the cases. Finally, we have designed an IFS on the dual of the real projective plane $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ and posed an open problem.

Chapter 5

Non-Affine Fractal Function on the Real Projective Plane

Approximation of a function is a useful concept in mathematics (13; 30). Stone-Weierstrass shows that a continuous real-valued function on a compact interval can be uniformly approximated by polynomials. In the last century, the approximation theory has been intensively studied in fractal geometry (15; 19; 25; 27; 28; 29). In our earlier research (22), we presented the concept of a projective FIF on the real projective plane. Following that, we proceeded to ascertain the fractal dimension of the graph associated with a certain projective FIF (21).
In this chapter, we mainly investigate the non-affine real projective fractal function (RPFF) on the real projective interval. Also, we answer the question posed in Ch... but for non-affine settings.

5.1 Approximation on real projective plane

In this section, we mainly, prove results concerning approximations of functions on the real projective plane.
The decomposition of the projective plane is essential for subsequent sections in order to establish and prove the results. In Chapter 3, section 3.1 we already provided a setup for the decomposition of the space $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. That is
$\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ can be expressed as

$$\mathbb{RP}^2 \setminus \mathbb{H}_{e_3} = \mathbb{H} \oplus \mathbb{K}.$$

Here, we use notations $\mathbb{H}$ for $\mathbb{H}_{10}$ and $\mathbb{K}$ for $\mathbb{H}_{01}$.

5.1.1 Projective polynomial approximation on projective intervals

For $n \geq 0$, the n^{th} degree projective polynomial $p_n : \mathbb{H} \to \mathbb{K}$ is defined by

$$p_n(x : 0 : z) = (0 : a_n x^n + a_{n-1} x^{n-1} z + \cdots + a_1 x^1 z^{n-1} + a_0 z^n : z^n),$$

where $a_i \in \mathbb{R}$ and $a_n \neq 0$. Since $p_n(\lambda x : 0 : \lambda z) = p_n(x : 0 : z)$ for all $\lambda \neq 0$. So, p_n's are well defined. The projective polynomial p_n can also be expressed as

$$p_n(x : 0 : z) = \big(a_n \odot (0 : x^n : z^n)\big) \oplus \big(a_{n-1} \odot (0 : x^{n-1} : z^{n-1})\big) \oplus \cdots \oplus \big(a_0 \odot (0 : 1 : 1)\big).$$

Let $\mathscr{P}_n(\mathbb{H})$ denotes the set of all projective polynomials of degree less than or equal to n, and let $\mathscr{P}(\mathbb{H}) := \bigcup_{n=1}^{\infty} \mathscr{P}_n(\mathbb{H})$.

Theorem 5.1.1 (Stone-Weierstrass type theorem) *Let* $\mathbb{P}_{\mathbb{I}\times\{0\}} := \big\{(x:0:z) \in \mathbb{H} :\ (0:0:1) \preceq (x:0:z) \preceq (1:0:1)\big\}$ *and* $f : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{K}$ *be a continuous function. Then there exists a sequence of projective polynomials* $p_n : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{K}$ *such that* $p_n \to f$ *uniformly.*

5.2 Non-affine real projective fractal function

In this section, mainly we introduce a non-affine fractal interpolation scheme to construct a non-affine fractal function on the real projective plane.

5.2.1 Construction of non-affine fractal function

Here, we provide the details of the construction of non-affine fractal functions. Let $\Delta_{\mathbb{P}} := \big\{(x_n : y_n : z_n) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : n = 0, 1, \ldots, N\big\}$ be the given set of points in $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ satisfying the conditions $(x_0 : 0 : z_0) \prec (x_1 : 0 : z_1) \prec \cdots \prec (x_N : 0 : z_N)$. With a slight abuse of notation, we can write

$$\Delta_{\mathbb{P}} = \big\{(x_n : y_n : z_n) = (x_n : 0 : z_n) \oplus (0 : y_n : z_n) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : n = 0, 1, \ldots, N\big\}.$$

Let $L_n : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{P}_{\mathbb{I}_n\times\{0\}}$, for $n = 1, 2, \ldots, N$, be the contractive homeomorphisms given in section 3.2. Let $g : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{K}$ be a continuous function such that $g(x_n : 0 : z_n) = (0 : y_n : z_n)$, for $n = 0, 1, \ldots, N$. Consider the continuous function $F_n : \mathbb{P}_{\mathbb{I}\times\{0\}} \oplus \mathbb{K} \to \mathbb{K}$,

$$F_n(x : y : z) = \alpha_n \odot (0 : y : z) \oplus g \circ L_n(x : 0 : z) \ominus \big(\alpha_n \odot b(x : 0 : z)\big), \tag{5.1}$$

where α_n's are scaling factors lying in $(-1, 1)$, $n = 0, 1, \ldots, N$ and $b : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{K}$ is the continuous function such that $b(x_0 : 0 : z_0) = g(x_0 : 0 : z_0)$, $b(x_N : 0 : z_N) = g(x_N : 0 : z_N)$ with $b \neq g$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1, 1)^N$ is the scale vector. Now, define the maps $W_n : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \to \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ by

$$W_n(x : y : z) = L_n(x : 0 : z) \oplus F_n(x : y : z). \tag{5.2}$$

Then $\big\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n :\ n = 1, 2, \ldots, N\big\}$ is an hyperbolic RPIFS on the real projective plane $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. Hence the RPIFS has an attractor G, say. Now, we will prove that G is the graph of a fractal function from $\mathbb{P}_{\mathbb{I}\times\{0\}}$ to $\mathbb{K}$. Let

$$\mathscr{G} := \big\{f \in \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}] :\ f(x_0 : 0 : z_0) = (0 : y_0 : z_0),\ f(x_N : 0 : z_N) = (0 : y_N : z_N)\big\}.$$

Then $\mathscr{G}$ is a closed subset of $\big(\mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}], \|\cdot\|_{\mathbb{P}\infty}\big)$. So, $\mathscr{G}$ is complete. Define a **real projective Read-Bajraktarević** operator (RPRB) $\mathcal{T} : \mathscr{G} \to \mathscr{G}$ such that

$$\mathcal{T}f(x : 0 : z) = F_n\left(L_n^{-1}(x : 0 : z) \oplus f \circ L_n^{-1}(x : 0 : z)\right). \tag{5.3}$$

Now, from (5.1) and (5.3), $\mathcal{T}$ can be written as

$$\mathcal{T}f(x : 0 : z) = g(x : 0 : z) \oplus \alpha_n \odot (f \ominus b) \circ L_n^{-1}(x : 0 : z). \tag{5.4}$$

Theorem 5.2.1 *The RPRB-operator $\mathcal{T}$ is well defined.*

Theorem 5.2.2 *The RPRB-operator $\mathcal{T}$ is contractive on $\left(\mathscr{G}, \|\cdot\|_{\mathbb{P}\infty}\right)$.*

Since $\left(\mathscr{G}, \|\cdot\|_{\mathbb{P}\infty}\right)$ is complete, by Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point g^α in $\mathscr{G}$. We call g^α as the non-affine **real projective fractal function** (RPFF). From (5.4), g^α satisfies the self-referential equation

$$g^\alpha(x:0:z) = g(x:0:z) \oplus \left(\alpha_n \odot (g^\alpha \ominus b) \circ L_n^{-1}(x:0:z)\right) \tag{5.5}$$

whenever $(x:0:z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$.

Theorem 5.2.3 *The graph of the non-affine RPFF g^α is the attractor of the RPIFS defined in (5.2). That is graph$(g^\alpha) = G$.*

The graphs of the classical function g and its corresponding non-affine RPFF g^α are given in Figure 5.1.

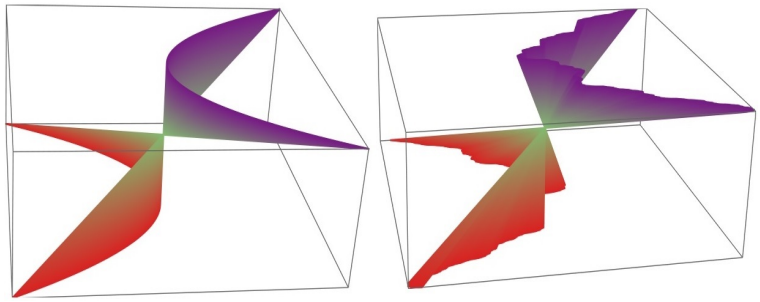

Figure 5.1: Graphs of the classical function (left) and corresponding non-affine RPFF (right).

5.2.2 Projective fractal operator

For a fixed partition $\Delta_{\mathbb{P}}$ and continuous function b, define

$$\mathscr{P}^\alpha_{\Delta_{\mathbb{P}},b} : \mathscr{G} \to \mathscr{G}$$

such that $\mathscr{P}^\alpha_{\Delta_{\mathbb{P}},b}(g) = g^\alpha$. We call the operator $\mathscr{P}^\alpha_{\Delta_{\mathbb{P}},b}$ as the **projective fractal operator**.

Theorem 5.2.4 *For fixed $\Delta_{\mathbb{P}}$ and b, $\mathscr{P}^\alpha$ satisfies the Lipschitz condition*

$$\|\mathscr{P}^\alpha(g) \ominus \mathscr{P}^\alpha(h)\|_{\mathbb{P}\infty} \leq \frac{1}{1-\alpha_\infty}\|g \ominus h\|_{\mathbb{P}\infty}.$$

Here onward, we consider the general case

$$b = \mathbb{L}(f), \tag{5.6}$$

where $\mathbb{L} : \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}] \to \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$ is a bounded linear operator.

Definition 5.2.1 *Let g^α be the non-affine RPFF function given by the IFS (5.2), (5.1) and (5.6) and the function associated to g with respect to the operator $\mathbb{L}$ and the partition $\Delta_{\mathbb{P}}$.*

Definition 5.2.2 *The fractal analogous of a projective polynomial $p(x:0:z) \in \mathscr{P}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$ is defined as $\mathscr{F}^{\alpha}(p) = p^{\alpha}$. Then we denote $\mathscr{P}_n^{\alpha}[\mathbb{P}_{\mathbb{I}\times\{0\}}] = \mathscr{F}^{\alpha}\left(\mathscr{P}_n[\mathbb{P}_{\mathbb{I}\times\{0\}}]\right)$ and $\mathscr{P}^{\alpha}[\mathbb{P}_{\mathbb{I}\times\{0\}}] = \mathscr{F}^{\alpha}\left(\mathscr{P}[\mathbb{P}_{\mathbb{I}\times\{0\}}]\right)$.*

Theorem 5.2.5 (Fractal version of Stone-Weierstrass type theorem) *Consider a continuous function $f : \mathbb{P}_{\mathbb{I}\times\{0\}} \to \mathbb{K}$. For a given $\epsilon > 0$, any partition $\Delta_{\mathbb{P}}$ of the projective interval $\mathbb{P}_{\mathbb{I}\times\{0\}}$ with N points, and for a bounded linear operator $\mathbb{L} : \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}] \to \mathscr{C}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$, $\mathbb{L} \neq Id$ satisfying $\mathbb{L}(f)(x_0:0:z_0) = f(x_0:0:z_0)$, $\mathbb{L}(f)(x_N:0:z_N) = f(x_N:0:z_N)$, there exists a fractal projective polynomial $p_0^{\alpha} \in \mathscr{P}^{\alpha}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$ with $\alpha_{\infty} \neq 0$, generated by the RPIFS (5.2), (5.1) and (5.6) such that $\|p_0^{\alpha} \ominus f\|_{\mathbb{P}\infty} < \epsilon$.*

Corollary 5.2.1 *The system $\left\{(0:1:1)^{\alpha}, (0:x:z)^{\alpha}, (0:x^2:z^2)^{\alpha}, \ldots, (0:x^m:z^m)^{\alpha}\right\}$ is linearly independent for any arbitrary scale vector α. In fact, the system forms a basis for $\mathscr{P}_m^{\alpha}[\mathbb{P}_{\mathbb{I}\times\{0\}}]$.*

5.3 Dual non-affine real projective fractal function

In this section, we first define the dual RPIFS but for non-affine settings and prove that the attractor of the dual RPIFS (corresponding to an RPIFS) is also the graph of a fractal function.
Recall, the space

$$\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} = \widehat{\mathbb{H}}_{10} \oplus \widehat{\mathbb{H}}_{01}. \tag{5.7}$$

We consider the maps

$$\widehat{L}_n : \widehat{\mathbb{P}}_{\mathbb{I}\times\{0\}} \to \widehat{\mathbb{P}}_{\mathbb{I}_n\times\{0\}}, \quad \widehat{F}_n : \widehat{\mathbb{P}}_{\mathbb{I}\times\{0\}} \oplus \widehat{\mathbb{H}}_{01} \to \widehat{\mathbb{H}}_{01}$$

which are defined in (4.4).Define $\widehat{W}_n : \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} \to \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ such that

$$\widehat{W}_n\big((x:y:z)^{\perp}\big) = \widehat{L}_n\big((x:0:z)^{\perp}\big) \oplus \widehat{F}_n\big((x:y:z)^{\perp}\big).$$

The collection $\left\{\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}; \widehat{W}_n : \ n = 1, 2, \ldots, N\right\}$ forms an contractive IFS on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$, which we will call as **dual RPIFS**. Hence it has a unique attractor $\widehat{G}$ on $\mathscr{H}\left(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}\right)$.

Theorem 5.3.1 *There exists a continuous function $\widehat{g}^{\alpha} : \widehat{\mathbb{P}}_{I\times\{0\}} \to \widehat{\mathbb{H}}_{01}$ such that graph$(\widehat{g}^{\alpha}) = \widehat{G}$. Moreover $\widehat{G} = \left\{p^{\perp} : \ p \in G\right\}$, where G is the attractor of the RPIFS defined in (5.2).*

The graphs of the classical dual function $\widehat{g}$ and its corresponding dual non-affine RPFF $\widehat{g}^{\alpha}$ are given in Figure 5.2.

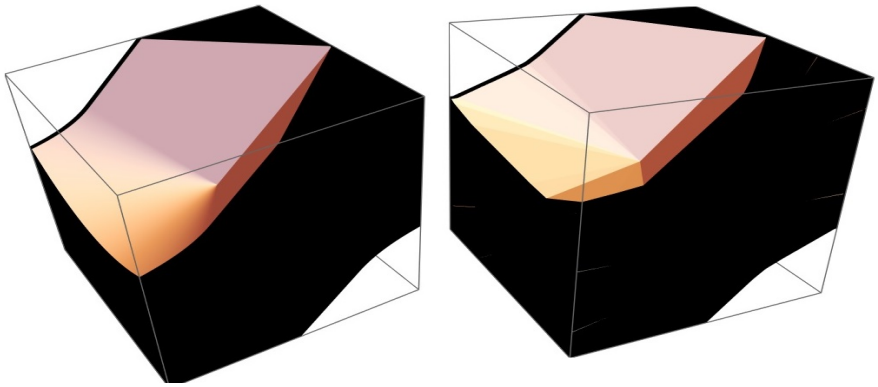

Figure 5.2: Graphs of the classical dual function (left) and corresponding dual non-affine RPFF (right).

Chapter 6

Quantization of Cantor-like set on the Projective Line

6.1 Projective line

The projective line, which is denoted by $\mathbb{RP}^1$, is the quotient of the set $\mathbb{R}^2 \setminus \{(0,0)\}$ of non-zero vectors by the equivalence relation "$x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}^*$" (non-zero reals). Now, for $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{\theta\}$, we denoted $[x] = [x_1 : x_2]$ is the equivalence class that containing $x = (x_1, x_2)$. The projective line may be identified with the line $\mathbb{R}$ extended by a point at infinity. More precisely, the line $\mathbb{R}$ may be identified with the subset of $\mathbb{RP}^1$ given by

$$\mathbb{RP}^* = \{[x : 1] : \ x \in \mathbb{R}\}.$$

This subset covers all points of $\mathbb{RP}^1$ except one, which is called the point at infinity, $\infty := [0 : 1]$. Thus

$$\mathbb{RP}^1 = \mathbb{RP}^* \cup \{\infty\}.$$

For $[x_1 : 1], [x_2 : 1] \in \mathbb{RP}^*$, define

$$[x_1 : 1] \oplus [x_2 : 1] = [x_1 + x_2 : 1]$$
$$[x_1 : 1][x_2 : 1] = [x_1 x_2 : 1]$$

and the scalar multiplication of an element $[x : 1] \in \mathbb{RP}^*$ with $c \in \mathbb{R}$ is defined by $c[x : 1] = [cx : 1]$. The difference between two elements $[x_1 : 1], [x_2 : 1] \in \mathbb{RP}^*$ is defined by

$$[x_1 : 1] \ominus [x_2 : 1] = [x_1 - x_2 : 1].$$

Definition 6.1.1 (Projective metric on $\mathbb{RP}^*$) *For $[x_1 : 1], [x_2 : 1] \in \mathbb{RP}^*$, define a distance function $d_{\mathbb{P}}$ on $\mathbb{RP}^*$ as follows:*

$$d_{\mathbb{P}}\big([x_1 : 1], [x_2 : 1]\big) := |x_1 - x_2|. \tag{6.1}$$

Definition 6.1.2 (Generating cone) *Let $[x], [y] \in \mathbb{RP}^*$. Then the point $[x], [y]$ generates two line segments, one of them does not intersect the point at infinity $[1 : 0]$, which is denoted by $C_{\overline{xy}}$, we are calling as* **cone generated by** *$[x], [y]$. A cone C is said to be* **multi-cone** *if C is the disjoint union of finite numbers of generating cones.*

Theorem 6.1.1 *Let* $\mathcal{P} = \left\{ A_1 = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 1 \end{pmatrix} \right\} \subset GL(2,\mathbb{R})$. *Then the associated RPIFS* $\mathscr{W}_{\mathcal{P}} = \left\{ \mathbb{RP}^1; w_{A_1}, w_{A_2} \right\}$, *where* $w_A : \mathbb{RP}^1 \to \mathbb{RP}^1$ *is given by* $w_A[x] = [Ax]$, *has an attractor.*

Theorem 6.1.2 *If* $F_{\mathcal{P}}$ *is the attractor of the RPIFS* $\mathscr{W}_{\mathcal{P}}$, *then* $\dim_H(F_{\mathcal{P}}) = \frac{\log 2}{\log 3}$.

6.2 Quantization error

Definition 6.2.1 *Let* μ *be the probability distribution on* $\mathbb{RP}^*$. *Then the* n*-th quantization error for* μ *of order* r *is defined by*

$$\mathbb{V}_{n,r}(\mu) = \inf_{\substack{\Delta \subset \mathbb{RP}^* \\ |\Delta| < n}} \int_{\mathbb{RP}^*} \min_{[a]\in\Delta} d_{\mathbb{P}}\big([x],[a]\big)^r d\mu[x].$$

Theorem 6.2.1 *Let* $p = (\frac{1}{2}, \frac{1}{2})$ *be the probability vector and* $\mathcal{P} = \left\{ A_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ 0 & \sqrt{3} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{3} \end{pmatrix} \right\}$ *and let* $\mathscr{W}_{\mathcal{P}} = \left\{ \mathbb{RP}^1; w_{A_1}, w_{A_2} \right\}$ *be the associated RPIFS. Then there exists unique probability measure* P *on* $\mathbb{RP}^1$ *such that*

$$P = \frac{1}{2} P \circ w_{A_1}^{-1} + \frac{1}{2} P \circ w_{A_2}^{-1}. \tag{6.2}$$

Now, if μ *is the Bernoulli measure on the sequence* $\{1,2\}^{\mathbb{N}}$ *with associated probability vector* $p = (\frac{1}{2}, \frac{1}{2})$, *then*

$$P(E) = \mu \circ \Pi^{-1}(E) \quad \text{for } E \subset \mathbb{RP}^*, \tag{6.3}$$

where $\Pi : \{1,2\}^{\mathbb{N}} \to F_{\mathcal{P}}$ *given by* $\Pi(\mathbf{i}) = \lim_{n\to\infty} w_{A_{i_1}} w_{A_{i_2}} \cdots w_{A_{i_n}}([x])$, *for arbitrary* $[x] \in \mathbb{RP}^1$.

Let $C_{\overline{ab}}$, be the cone generated by the points $[a] = [-1 : 1]$ and $[b] = [1 : 1]$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_l) \in \{1,2\}^l$, define $w_{A_\omega} = w_{A_{\omega_1}} \circ w_{A_{\omega_2}} \circ \cdots \circ w_{A_{\omega_l}}$. Let $C_\omega = w_{A_\omega}(C_{\overline{ab}})$ and $[c]$, $[d]$ be the boundary points of the cone C_ω. Let $m_\omega \in C_\omega$ be the point which lies in the equal distance from the points $[c]$ and $[d]$ with respect to the metric $d_{\mathbb{P}}$. The sets $(C_\omega)_{\omega\in\{1,2\}^l}$ consists 2^l cone in the l-th level of the projective Cantor set construction. The cones $C_{\omega 1^*}$, $C_{\omega 2^*}$ are said to be children of C_ω into which C_ω is split up at $l+1$-th level. Let $m_{\omega 1^*} \in C_{\omega 1^*}$ and $m_{\omega 2^*} \in C_{\omega 2^*}$ be the same points as m_ω. Let $F_{\mathbb{P}} = \bigcap_{l\in\mathbb{N}} \bigcup_{\omega\in\{1,2\}^l} C_\omega$. We call it **projective Cantor set**.

Let $n \in \mathbb{N}$ with $n \geq 1$, and $k(n) \in \mathbb{N}$ be the unique number such that $2^{k(n)} \leq n < 2^{k(n)+1}$. Let $\mathbb{I} \subset \{1,2\}^{k(n)}$ with $card(\mathbb{I}) = n - 2^{k(n)}$, and let Δ_n be the set consisting af all points m_ω of cone C_ω with $\omega \in \{1,2\}^{k(n)} \setminus \mathbb{I}$ and all points $m_{\omega 1^*}$, $m_{\omega 2^*}$ of the children of C_ω with $\omega \in \mathbb{I}$. That is

$$\Delta_n = \left\{ m_\omega : \omega \in \{1,2\}^{k(n)} \setminus \mathbb{I} \right\} \cup \left\{ m_{\omega 1^*} : \omega \in \mathbb{I} \right\} \cup \left\{ m_{\omega 2^*} : \omega \in \mathbb{I} \right\}.$$

Let $D_n = \frac{1}{2} \frac{1}{18^{k(n)}} \left[2^{k(n)+1} - n + \frac{1}{9}(n - 2^{k(n)}) \right]$. Then we get the following:

Theorem 6.2.2 *The upper bound of the* n*-th quantization error for* P *of order* 2 *is* D_n.

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